$$= x_{d_{h'}} + \sum_{k=1}^{n-r} S_{h'k} \cdot x_{f_k}.$$

$\therefore$ for any $h = 1, 2, \cdots, r$, we have

$$x_{d_h} = -\sum_{k=1}^{n-r} S_{hk}\, x_{f_k}.$$

Since
$$\left(\sum_{k=1}^{n-r} x_{f_k} \cdot u_k\right)_l = \sum_{k=1}^{n-r} x_{f_k} \cdot u_{kl}$$

if $l = d_h$, $\displaystyle = -\sum_{k=1}^{n-r} x_{f_k} \cdot S_{hk} = x_{d_h} = x_l.$

if $l = f_{k'}$, $\displaystyle = \sum_{k=1}^{n-r} x_{f_k} \cdot u_{kf_{k'}} = x_{f_{k'}} = x_l.$

$$\boxed{\therefore \quad \sum_{k=1}^{n-r} x_{f_k} \cdot u_k = x.}$$

Week 9 :

Example : $A = \begin{bmatrix} 1 & 4 & 0 & -1 & 0 & 7 & -9 \\ 2 & 8 & -1 & 3 & 9 & -13 & 7 \\ 0 & 0 & 2 & -3 & -4 & 12 & -8 \\ 1 & -4 & 2 & 4 & 8 & -31 & 37 \end{bmatrix}$. find Null(A).

$$\xrightarrow{\text{row op.}} A' (= \text{RREF}) = \begin{array}{c} \begin{bmatrix} 1 & 4 & 0 & 0 & 2 & 1 & -3 \\ 0 & 0 & 1 & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 1 & 2 & -6 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccccc} d_1 & f_1 & d_2 & d_3 & f_2 & f_3 & f_4 \end{array} \end{array}$$

$\therefore \text{Null}(A) = \text{span}\{u_1, u_2, u_3, u_4\}$ where

$$u_1 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad u_2 = \begin{bmatrix} -2 \\ 0 \\ -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad u_3 = \begin{bmatrix} -1 \\ 0 \\ 3 \\ 6 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad u_4 = \begin{bmatrix} 3 \\ 0 \\ -5 \\ -6 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

<u>next question</u>: Given $S = \{u_1, u_2, \cdots, u_n\} \subseteq \mathbb{R}^m$, how to find $S' \subseteq S$ s.t. $\text{span}(S) = \text{span}(S')$??

<u>Thm</u>: Let $A = [u_1\ u_2 \cdots u_n]$ be a $m \times n$ matrix and A' be its RREF. Suppose the pivot column is given by the $d_1, d_2, \cdots, d_r$ th column of A' where $\text{rank}(A) = r \neq 0$. Then $\{u_{d_1}, \cdots, u_{d_r}\} = S'$ is a basis of $\text{span}(S)$.

Moreover if $u_j' = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_r \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, then $u_j = \sum_{i=1}^{r} \alpha_i u_{d_i}$

<u>Example</u>: $\left\{ \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \right\} = S$

$\quad\ \ u_1 \quad\ \ u_2 \quad\ \ u_3 \quad\ \ u_4 \quad\ \ u_5$

$$A = \begin{bmatrix} 0 & 1 & 1 & 2 & 2 \\ 1 & 2 & 3 & 2 & 3 \\ 2 & -1 & -3 & 3 & 1 \end{bmatrix} \longrightarrow \begin{array}{ccccc} & & f_1 & & f_2 \end{array} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = \text{RREF}.$$

$\therefore \text{rank}(A) = 3 \quad \therefore \text{Span}(S) = \text{span}\{u_1, u_2, u_4\}$

while $\begin{cases} u_3' = u_1' + u_2' \Rightarrow u_3 = u_1 + u_2 \\ u_5' = u_1' + u_3' \Rightarrow u_5 = u_1 + u_3 \end{cases}$ #

Eg 2:

$$S = \left\{ \begin{array}{c} u_1 \\ \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix} \end{array}, \begin{array}{c} u_2 \\ \begin{bmatrix} 2 \\ 1 \\ 2 \\ -1 \end{bmatrix} \end{array}, \begin{array}{c} u_3 \\ \begin{bmatrix} 7 \\ 3 \\ 5 \\ -5 \end{bmatrix} \end{array}, \begin{array}{c} u_4 \\ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \end{bmatrix} \end{array}, \begin{array}{c} u_5 \\ \begin{bmatrix} -1 \\ 0 \\ 9 \\ 0 \end{bmatrix} \end{array} \right\}$$

$$A = \begin{bmatrix} 1 & 2 & 7 & 1 & -1 \\ 1 & 1 & 3 & 1 & 0 \\ 3 & 2 & 5 & -1 & 9 \\ 1 & -1 & -5 & 2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & 0 & 3 \\ 0 & 1 & 4 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore \operatorname{Span}(S) = \operatorname{Span}\{u_1, u_2, u_4\}$

And $\begin{cases} u_3 = -u_1 + 4u_2 \quad (\because u_3' = -u_1' + 4u_2') \\ u_5 = 3u_1 - u_2 - 2u_4 \quad (\because u_5' = 3u_1' - u_2' - 2u_4') \end{cases}$

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proof of thm:

$\because A' = \text{RREF of } A$

$\therefore \exists$ non-singular $m \times m$ matrix H s.t. $HA = A'$

(given by product of row op. matrix)

$\therefore u_j' = A' e_j = HAe_j = H u_j, \ \forall j = 1, 2, \cdots, n$

[Recall a thm: If $\{Av_1, \cdots, Av_n\}$ is linearly independent,

then $\{v_1, \cdots, v_n\}$ is linearly independent.]

$\because$ $u'_{d_j} = e_j \Rightarrow \{u'_{d_1}, \cdots, u'_{d_r}\}$ is linearly indep.

$\Rightarrow \{u_{d_1}, \cdots, u_{d_r}\}$ is linearly indep.

[It remains to show $\text{span}(S) \subseteq \text{span}\{u_{d_1}, \cdots, u_{d_r}\}$.

It suffices to show that $u_{f_j} \in \text{span}\{u_{d_1}, \cdots, u_{d_r}\}$, $\forall j = 1, 2, \ldots, n-r$

By defn of free column,

$$u'_{f_j} = \sum_{i=1}^{r} \alpha_{ij} \underbrace{u'_{d_i}}_{= e_i} \quad \text{for some } \alpha_{ij} \in \mathbb{R}.$$

(Since $H u_k = u'_k$ for all k)

$$H u_{f_j} = \sum_{i=1}^{r} \alpha_{ij} H \cdot u_{d_i}$$

$\because$ H is invertible $\therefore$ $u_{f_j} = \sum_{i=1}^{r} \alpha_{ij} u_{d_i} \in \text{span}\{u_{d_1}, \cdots, u_{d_r}\}$ 犇